Mathematics Content

MORNING SESSION (120 questions in 12 topic areas)

Topic Area	Approximate Percentage of Test Content
I. Mathematics	**15%**
A. Analytic geometry	
B. Integral calculus	
C. Matrix operations	
D. Roots of equations	
E. Vector analysis	
F. Differential equations	
G. Differential calculus	

- Approximately 18 questions
- Total time: 4 hours = 2 minutes/question
- Examples today from *1001 Solved Engineering Fundamentals Problems* – Michael R. Lindeburg PE (Professional Publications, CA)

Mathematics Overview

- Be familiar with material included in handbook (especially terminology)
- Calculators allowed:
  - Hewlett Packard - HP 33s, HP 35s, ~~HP 9s~~
  - Casio – any FX 115
  - Texas Instruments – any TI 30X, any TI 36X

  (next update: Nov. 15, 2012)

Tips!

1. Read the question
2. Use multiple choice
3. $e \approx 3$, $\pi \approx 3$
4. Be familiar with presentation of material in the handbook

   (see http://www.ncees.org/Exams/Study_materials/Download_FE_Supplied-Reference_Handbook.php or google "fe handbook" and click the first link)

Tips!

1. Read the question
2. Use multiple choice
3. $e \approx 3$, $\pi \approx 3$
4. Be familiar with presentation of material in the handbook
5. Expect "knowledge" based questions

Naperian logarithms have a base closest to which number?

(A) 2.17 (B) 2.72 (C) 3.14 (D) 10.0

Fundamentals of Engineering: Mathematics Review

TIP 1: Read the Question

What is the solution of the equation $50x^2 + 5(x-2)^2 = -1$, where x is a real-valued variable?

(A) -6.12 and -3.88 (B) -0.52 and 0.700

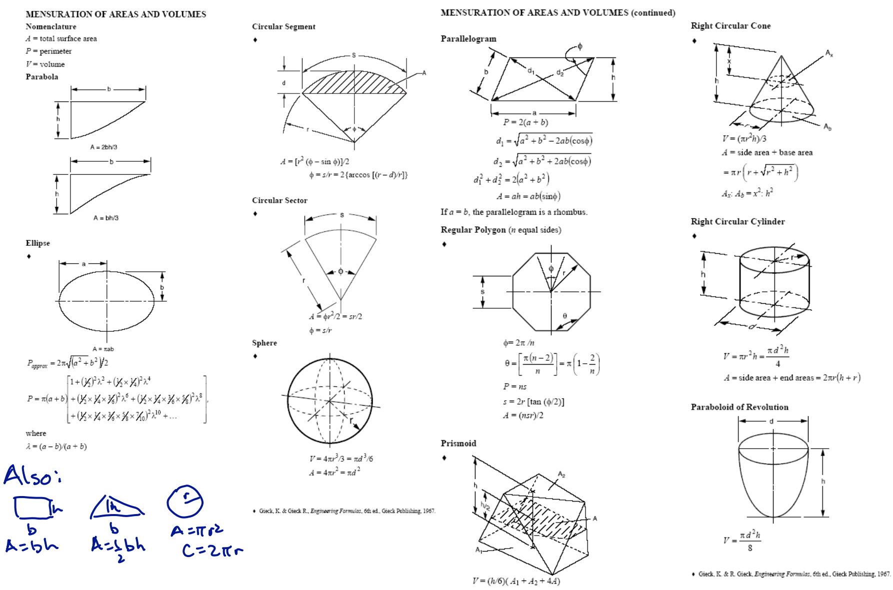
(C) 7.55 (D) no solution

TIP 2: Use the multiple choice

What are the roots of the cubic equation $x^3 - 8x - 3 = 0$?

(A) $x = -7.90, -3, -0.38$
(B) $x = -3, -2, 2$
(C) $x = -3, -0.38, 2$
(D) $x = -2.62, -0.38, 3$

Geometry: Area & Volume



Also:

b h $A = bh$

b h $A = \frac{1}{2}bh$

r $A = \pi r^2$ $C = 2\pi r$

Example: Volume

A cubical container that measures 2 m on a side is tightly packed with eight balls and is filled with water. All eight balls are in contact with the walls of the container and the adjacent balls. All of the balls are the same size. What is the volume of water in the container?

(A) 0.38 m^3 (B) 2.5 m^3 (C) 3.8 m^3 (D) 4.2 m^3

Geometry: Straight Lines

STRAIGHT LINE

The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

which is also known as the *slope-intercept* form.

The *point-slope* form is $y - y_1 = m(x - x_1)$

Given two points: slope, $m = (y_2 - y_1)/(x_2 - x_1)$

The angle between lines with slopes m_1 and m_2 is

$$\alpha = \arctan\,[(m_2 - m_1)/(1 + m_2 \cdot m_1)]$$

Two lines are perpendicular if $m_1 = -1/m_2$

The distance between two points is

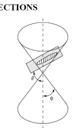
$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Plane Surface:
General eqn:
$Ax + By + Cz + D = 0$

The perpendicular distance, d, from point (x_3, y_3) to line $Ax+By+C=0$ is: $d = \dfrac{|Ax_3 + By_3 + C|}{\sqrt{A^2+B^2}}$

Geometry: Conic Sections and Quadric Surfaces

CONIC SECTIONS

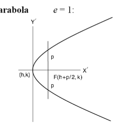


e = eccentricity = cos θ/(cos φ)

[Note: X' and Y', in the following cases, are translated axes.]

Case 1. Parabola $e = 1$:

♦

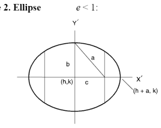


$$(y - k)^2 = 2p(x - h);\ \text{Center at } (h, k)$$

is the standard form of the equation. When $h = k = 0$, Focus: $(p/2, 0)$; Directrix: $x = -p/2$

Case 2. Ellipse $e < 1$:

♦



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1;\quad \text{Center at } (h,k)$$

is the standard form of the equation. When $h = k = 0$,

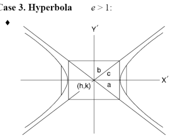
Eccentricity: $e = \sqrt{1 - (b^2/a^2)} = c/a$

$b = a\sqrt{1 - e^2};$

Focus: $(\pm ae, 0)$; Directrix: $x = \pm a/e$

Case 3. Hyperbola $e > 1$:

♦



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1;\quad \text{Center at } (h,k)$$

is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1 + (b^2/a^2)} = c/a$

$b = a\sqrt{e^2 - 1};$

Focus: $(\pm ae, 0)$; Directrix: $x = \pm a/e$

♦ Brink, R.W., *A First Year of College Mathematics*, D. Appleton-Century Co., Inc., 1937.

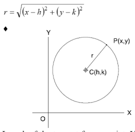
Case 4. Circle $e = 0$:

$$(x - h)^2 + (y - k)^2 = r^2;\qquad \text{Center at } (h, k)$$

is the general form of the equation with radius

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

♦

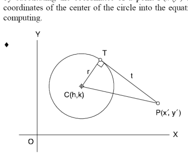


Length of the tangent from a point. Using the general form of the equation of a circle, the length of the tangent is found from

$$t^2 = (x' - h)^2 + (y' - k)^2 - r^2$$

by substituting the coordinates of a point $P(x', y')$ and the coordinates of the center of the circle into the equation and computing.

♦



Conic Section Equation

The general form of the conic section equation is

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

where not both A and C are zero.

If $B^2 - AC < 0$, an *ellipse* is defined.

If $B^2 - AC > 0$, a *hyperbola* is defined.

If $B^2 - AC = 0$, the conic is a *parabola*.

If $A = C$ and $B = 0$, a *circle* is defined.

If $A = B = C = 0$, a *straight line* is defined.

$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the normal form of the conic section equation, if that conic section has a principal axis parallel to a coordinate axis.

$$h = -a;\ k = -b$$

$$r = \sqrt{a^2 + b^2 - c}$$

If $a^2 + b^2 - c$ is positive, a *circle*, center $(-a, -b)$.

If $a^2 + b^2 - c$ equals zero, a *point* at $(-a, -b)$.

If $a^2 + b^2 - c$ is negative, locus is *imaginary*.

QUADRIC SURFACE (SPHERE)

The general form of the equation is

$$(x - h)^2 + (y - k)^2 + (z - m)^2 = r^2$$

with center at (h, k, m).

In a three-dimensional space, the distance between two points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

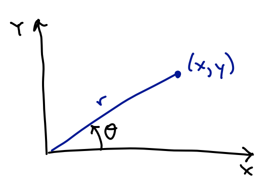
Geometry

What is the radius of the circle defined by $x^2 + y^2 - 4x + 8y = 7$?

(A) $\sqrt{3}$ (B) $2\sqrt{5}$ (C) $3\sqrt{3}$ (D) $4\sqrt{3}$

Geometry: Coordinate Systems

Polar



$x = r\cos\theta$

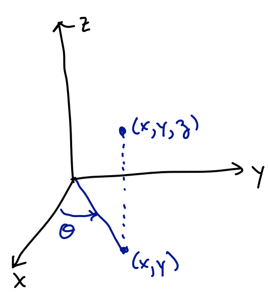
$y = r\sin\theta$

$\tan\theta = \dfrac{y}{x}$

(see p.5 of handbook)

*θ pos from x to y

Cylindrical



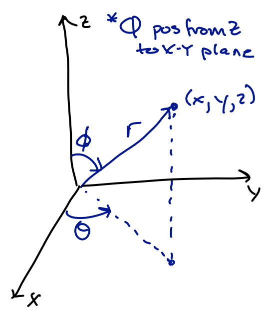
$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Spherical

*ϕ pos from z to x-y plane



$x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

University of Utah

Algebra: Partial Fractions

Given $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials:

want $\frac{P(x)}{Q(x)} = \frac{A(x)}{M(x)} + \frac{B(x)}{N(x)} + \frac{C(x)}{O(x)} \ldots$

① $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ [n different linear terms]

$\rightarrow \frac{P(x)}{Q(x)} = \sum_{i=1}^{n} \frac{A_i}{x-a_i}$ $\quad \frac{P(x)}{Q(x)} = \frac{5}{(x+3)(x-3)} \overset{\text{WANT:}}{=} \frac{A}{x+3} + \frac{B}{x-3}$

② $Q(x) = (x-a)^n$ [n identical linear terms]

$\rightarrow \frac{P(x)}{Q(x)} = \sum_{i=1}^{n} \frac{A_i}{(x-a)^i}$ $\quad \frac{P(x)}{Q(x)} = \frac{5}{(x+3)^2} = \frac{A}{(x+3)} + \frac{B}{(x+3)^2}$

③ $Q(x) = (x^2+a_1x+b_1)(x^2+a_2x+b_2)\cdots(x^2+a_nx+b_n)$ [n different quadratic terms]

$\rightarrow \frac{P(x)}{Q(x)} = \sum_{i=1}^{n} \frac{A_ix+B_i}{x^2+a_ix+b_i}$

④ $Q(x) = (x^2+ax+b)^n$ [n identical quad. terms]

$\rightarrow \frac{P(x)}{Q(x)} = \sum_{i=1}$

⑤ combination of ①-④

Add appropriate expansions.

Algebra: Partial Fractions

Example: $\frac{5}{x^2+3x+2}$

TIP 2: Use multiple choice

Express $\dfrac{4}{x^2(x^2-4x+4)}$ as the sum of fractions.

a) $\dfrac{1}{x}-\dfrac{1}{x-2}+\dfrac{1}{(x-2)^2}$

b) $\dfrac{1}{x}+\dfrac{1}{x^2}-\dfrac{1}{x-2}+\dfrac{1}{(x-2)^2}$

c) $\dfrac{1}{x^2}+\dfrac{1}{(x-2)^2}$

d) $\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x-2}+\dfrac{1}{(x-2)^2}$

Algebra: Exponents & Logarithms

Exponents

For $x, y > 0$

$x^{-a} = \dfrac{1}{x^a}$

$x^a x^b = x^{a+b}$

$(xy)^a = x^a y^a$

$x^{ab} = (x^a)^b$

LOGARITHMS

The logarithm of x to the Base b is defined by

$\log_b(x) = c$, where $b^c = x$

Special definitions for $b = e$ or $b = 10$ are:

$\ln x$, Base $= e$

$\log x$, Base $= 10$

To change from one Base to another:

$\log_b x = (\log_a x)/(\log_a b)$

e.g., $\ln x = (\log_{10} x)/(\log_{10} e) = 2.302585\ (\log_{10} x)$

Identities

$\log_b b^n = n$

$\log x^c = c \log x$; $x^c = \text{antilog}\,(c \log x)$

$\log xy = \log x + \log y$

$\log_b b = 1$; $\log 1 = 0$; $\log A < 0,\ 0 < A < 1$; $> 0,\ A > 1$

$\log x/y = \log x - \log y$

- Brink, R.W., *A First Year of College Mathematics*, D. Appleton-Century Co., Inc., Englewood Cliffs, NJ, 1937.

TIP 3: e ≈ 3 (π ≈ 3, too)

A growth curve is given by $A = 10\,e^{2t}$. At what value of t is A = 100?

a) 5.261 b) 3.070 c) 1.151 d) 0.726

If $\ln x = 3.2$, what is x?

a) 18.65 b) 24.53 c) 31.83 d) 64.58

Trigonometry: Functions

TRIGONOMETRY

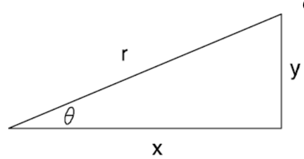
Trigonometric functions are defined using a right triangle.

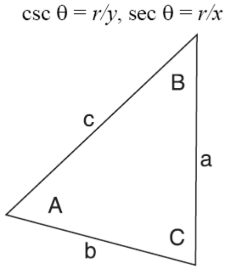
$\sin\theta = y/r$, $\cos\theta = x/r$

$\tan\theta = y/x$, $\cot\theta = x/y$

$\csc\theta = r/y$, $\sec\theta = r/x$

	0	30°	45°	60°	90°
sin	0	½	√2/2	√3/2	1
cos	1	√3/2	√2/2	1/2	0
tan	0	1/√3	1	√3	∞





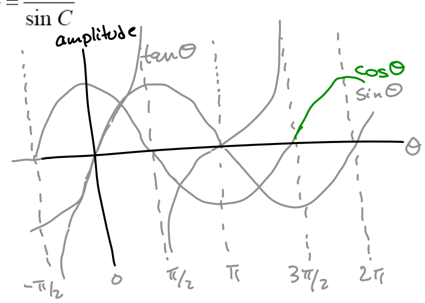
Law of Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos A$

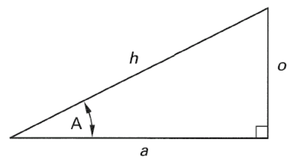
$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$



Example: Trig

If the sine of angle A is given as K, what is the tangent of angle A?



(A) $\frac{hK}{o}$ (B) $\frac{aK}{h}$

(C) $\frac{ha}{K}$ (D) $\frac{hK}{a}$

Trigonometry: Identities

Identities

$\csc\theta = 1/\sin\theta$

$\sec\theta = 1/\cos\theta$

$\tan\theta = \sin\theta/\cos\theta$

$\cot\theta = 1/\tan\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

$\tan 2\alpha = (2\tan\alpha)/(1-\tan^2\alpha)$

$\cot 2\alpha = (\cot^2\alpha - 1)/(2\cot\alpha)$

$\tan(\alpha+\beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$

$\cot(\alpha+\beta) = (\cot\alpha\cot\beta - 1)/(\cot\alpha + \cot\beta)$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\tan(\alpha-\beta) = (\tan\alpha - \tan\beta)/(1 + \tan\alpha\tan\beta)$

$\cot(\alpha-\beta) = (\cot\alpha\cot\beta + 1)/(\cot\beta - \cot\alpha)$

$\sin(\alpha/2) = \pm\sqrt{(1-\cos\alpha)/2}$

$\cos(\alpha/2) = \pm\sqrt{(1+\cos\alpha)/2}$

$\tan(\alpha/2) = \pm\sqrt{(1-\cos\alpha)/(1+\cos\alpha)}$

$\cot(\alpha/2) = \pm\sqrt{(1+\cos\alpha)/(1-\cos\alpha)}$

$\sin\alpha\sin\beta = (1/2)[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$\cos\alpha\cos\beta = (1/2)[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

$\sin\alpha\cos\beta = (1/2)[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\sin\alpha + \sin\beta = 2\sin(1/2)(\alpha+\beta)\cos(1/2)(\alpha-\beta)$

$\sin\alpha - \sin\beta = 2\cos(1/2)(\alpha+\beta)\sin(1/2)(\alpha-\beta)$

$\cos\alpha + \cos\beta = 2\cos(1/2)(\alpha+\beta)\cos(1/2)(\alpha-\beta)$

$\cos\alpha - \cos\beta = -2\sin(1/2)(\alpha+\beta)\sin(1/2)(\alpha-\beta)$

hyperbolic: $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\tanh x = \frac{\sinh x}{\cosh x}$

Example: Trigonometry

What is an equivalent expression for $\sin 2x$?

(A) $\frac{1}{2}\sin x \cos x$ (B) $2\sin x \cos \frac{1}{2}x$ (C) $-2\sin x \cos x$ (D) $\dfrac{2\sin x}{\sec x}$

Calculus: Taylor's Series

Taylor's Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

is called *Taylor's series*, and the function $f(x)$ is said to be expanded about the point a in a Taylor's series.

If $a = 0$, the Taylor's series equation becomes a *Maclaurin's series*.

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

$\frac{1}{1-x} = 1 + x + x^2 + \ldots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$

The Taylor series expansion for $\cos x$ contains which powers of x?

(A) 0, 2, 4, 6, 8, ...
(B) 1, 3, 5, 9, ...
(C) 1, 2, 3, 4, 5, ...
(D) 1/2, 3/2, 5/2, 7/2, ...

Example: Taylor's Series / L'Hôpital's

L'Hospital's Rule (L'Hôpital's Rule)

If the fractional function $f(x)/g(x)$ assumes one of the indeterminate forms 0/0 or ∞/∞ (where α is finite or infinite), then

$$\lim_{x\to\alpha} f(x)/g(x)$$

is equal to the first of the expressions

$$\lim_{x\to\alpha}\frac{f'(x)}{g'(x)},\ \lim_{x\to\alpha}\frac{f''(x)}{g''(x)},\ \lim_{x\to\alpha}\frac{f'''(x)}{g'''(x)}$$

which is not indeterminate, provided such first indicated limit exists.

Example: Find $\dfrac{\sin(x)}{x}$ as $x \to 0$

Compute the following limit.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

(A) 0 (B) 1/4 (C) 1/2 (D) 1

Calculus: Differentiation and Integration

DIFFERENTIAL CALCULUS

The Derivative

For any function $y = f(x)$,

the derivative $= D_x y = dy/dx = y'$

$$y' = \lim_{\Delta x\to 0}\left[(\Delta y)/(\Delta x)\right]$$

$$= \lim_{\Delta x\to 0}\{[f(x+\Delta x) - f(x)]/(\Delta x)\}$$

y' = the slope of the curve $f(x)$.

Test for a Maximum

$y = f(x)$ is a maximum for

$x = a$, if $f'(a) = 0$ and $f''(a) < 0$.

Test for a Minimum

$y = f(x)$ is a minimum for

$x = a$, if $f'(a) = 0$ and $f''(a) > 0$.

Test for a Point of Inflection

$y = f(x)$ has a point of inflection at $x = a$,

if $f''(a) = 0$, and

if $f''(x)$ changes sign as x increases through $x = a$.

The Partial Derivative

In a function of two independent variables x and y, a derivative with respect to one of the variables may be found if the other variable is *assumed* to remain constant. If y *is kept fixed*, the function

$$z = f(x, y)$$

becomes a function of the *single variable* x, and its derivative (if it exists) can be found. This derivative is called the *partial derivative of z with respect to x*. The partial derivative with respect to x is denoted as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x}$$

INTEGRAL CALCULUS

The definite integral is defined as:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,dx$$

Also, $\Delta x_i \to 0$ for all i.

Calculus: Integration by Parts

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Example: $\int x e^{x}\,dx$

Examples: Calculus

Find the slope of the line tangent to the curve $y = x^3 - 2x + 1$ at $x = 1$.

(A) 1/4 (B) 1/3 (C) 1/2 (D) 1

Determine the equation of the line tangent to the graph $y = 2x^2 + 1$ at the point (1,3).

(A) $y = 2x + 1$
(B) $y = 4x - 1$
(C) $y = 2x - 1$
(D) $y = 4x + 1$

Examples: Calculus

Find the second derivative of $y = \sqrt{x^2} + x^{-2}$.

(A) $1 - 2x^{-3}$ (B) $1 - 6x^{-4}$ (C) 3 (D) $\frac{6}{x^4}$

What is the maximum of the function $y = -x^3 + 3x$, for $x \geq -1$?

(A) -2 (B) -1 (C) 0 (D) 2

Differential Equations: First Order

First-Order Linear Homogeneous Differential Equations With Constant Coefficients

$y' + ay = 0$, where a is a real constant:

Solution, $y = Ce^{-at}$

where C = a constant that satisfies the initial conditions.

First-Order Linear Nonhomogeneous Differential Equations

$$\tau \frac{dy}{dt} + y = Kx(t) \qquad x(t) = \begin{Bmatrix} A & t < 0 \\ B & t > 0 \end{Bmatrix}$$

$$y(0) = KA$$

τ is the time constant

K is the gain

The solution is

$$y(t) = KA + (KB - KA)\left(1 - \exp\left(\frac{-t}{\tau}\right)\right) \quad \text{or}$$

$$\frac{t}{\tau} = \ln\left[\frac{KB - KA}{KB - y}\right]$$

Example:

* $y' + 2y = \underline{4x}$, given $y(0) = 2$

(1) Homogeneous

$y' + 2y = 0 \rightarrow y_h = Ce^{-2x}$

(2) Particular

i) "Say" $y_p = Ax + B$ ~~+cos~~

ii) Deriv: $y'_p = A$

iii) Substitute: $(A) + 2(Ax + B) = 4x$

$(A + 2B) + (2A - 4)x = 0$

$2A - 4 = 0 \Rightarrow A = 2$

$2 + 2B = 0 \rightarrow B = -1$

(3) Combine:

$y(x) = y_h(x) + y_p(x) = Ce^{-2x} + 2x - 1$

(4) Use I.C.:

$2 = Ce^{-2 \cdot 0} + 2 \cdot 0 - 1 \rightarrow C = 3$

(5) Combine:

$y = 3e^{-2x} + 2x - 1$

Differential Equations: Second Order

Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

An equation of the form

$$y'' + 2ay' + by = 0$$

can be solved by the method of undetermined coefficients where a solution of the form $y = Ce^{rx}$ is sought. Substitution of this solution gives

$$(r^2 + 2ar + b)\,Ce^{rx} = 0$$

and since Ce^{rx} cannot be zero, the characteristic equation must vanish or

$$r^2 + 2ar + b = 0$$

The roots of the characteristic equation are

$$r_{1,2} = -a \pm \sqrt{a^2 - b}$$

and can be real and distinct for $a^2 > b$, real and equal for $a^2 = b$, and complex for $a^2 < b$.

If $a^2 > b$, the solution is of the form (overdamped)

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $a^2 = b$, the solution is of the form (critically damped)

$$y = (C_1 + C_2 x)e^{r_1 x}$$

If $a^2 < b$, the solution is of the form (underdamped)

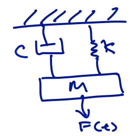
$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x), \text{ where}$$

$$\alpha = -a$$

$$\beta = \sqrt{b - a^2}$$

Example: Spring-Mass-Damper:

$M\ddot{y} + c\dot{y} + Ky = F(t)$



Find solution if $M=2$, $c=12$, $K=50$, $F(t)=60\sin 5t$

$2y'' + 12y' + 50y = 60\sin 5t$

char. eqn: let $y = e^{at}$

$\hookrightarrow 2r^2 + 12r + 50 = 0 \rightarrow r_{1,2} = -3 \pm 4i$

homog: $y_h = e^{-3t}(c_1 \sin 4t + c_2 \cos 4t)$

particular: $y_p(t) = A\sin 5t + B\cos 5t$

substitute $\Rightarrow$ MESS! $\Rightarrow B = -1,\ A = 0$

" $60\sin 5t$

solution:

$y(t) = (c_1 \sin 4t + c_2 \cos 4t)e^{-3t} - \cos 5t$

(use init. cond. for c_1, c_2)

Example: Differential Equations

What is a solution of the first-order difference equation $y(k+1) = y(k) + 5$?

(A) $y(k) = 4 - \dfrac{5}{k}$

(B) $y(k) = C - k$, where C is a constant

(C) $y(k) = 5^k + \dfrac{1 - 5^k}{-4}$

(D) $y(k) = 20 + 5k$

Example: Differential Equations

Determine the solution of the following differential equation.

$$y' + 5y = 0$$

(A) $y = 5x + C$ (B) $y = Ce^{-5x}$ (C) $y = Ce^{5x}$ (D) (A) or (B)

In the following differential equation with the initial condition $x(0) = 12$, what is the value of $x(2)$?

$$\frac{dx}{dt} + 4x = 0$$

(A) 3.35×10^{-4} (B) 4.03×10^{-3} (C) 3.35 (D) 6.04

Example: Differential Equations

What is the general solution of the following differential equation?

$$\frac{d^2y}{dx^2} + 4y = 0$$

(A) $y = \sin x + 2\tan x + C$
(B) $y = e^x - 2e^{-x} + C$
(C) $y = 2x^2 - x + C$
(D) $y = \sin 2x + \cos 2x + C$

Fundamentals of Engineering: Mathematics Review

Linear Algebra: Matrices

MATRICES

A matrix is an ordered rectangular array of numbers with m rows and n columns. The element a_{ij} refers to row i and column j.

Multiplication

If $\boldsymbol{A} = (a_{ik})$ is an $m \times n$ matrix and $\boldsymbol{B} = (b_{kj})$ is an $n \times s$ matrix, the matrix product AB is an $m \times s$ matrix

$$\boldsymbol{C} = \left(c_{ij}\right) = \left(\sum_{l=1}^{n} a_{il} b_{lj}\right)$$

where n is the common integer representing the number of columns of $\boldsymbol{A}$ and the number of rows of $\boldsymbol{B}$ (l and $k = 1, 2, \ldots, n$).

Addition

If $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ are two matrices of the same size $m \times n$, the sum $\boldsymbol{A} + \boldsymbol{B}$ is the $m \times n$ matrix $\boldsymbol{C} = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$.

Identity

The matrix $\mathbf{I} = (a_{ij})$ is a square $n \times n$ identity matrix where $a_{ii} = 1$ for $i = 1, 2, \ldots, n$ and $a_{ij} = 0$ for $i \neq j$.

Transpose

The matrix $\boldsymbol{B}$ is the transpose of the matrix $\boldsymbol{A}$ if each entry b_{ji} in $\boldsymbol{B}$ is the same as the entry a_{ij} in $\boldsymbol{A}$ and conversely. In equation form, the transpose is $\boldsymbol{B} = \boldsymbol{A}^T$.

Inverse

The inverse $\boldsymbol{B}$ of a square $n \times n$ matrix $\boldsymbol{A}$ is

$$\boldsymbol{B} = \boldsymbol{A}^{-1} = \frac{adj(\boldsymbol{A})}{|\boldsymbol{A}|}, \text{ where}$$

adj($\boldsymbol{A}$) = adjoint of $\boldsymbol{A}$ (obtained by replacing $\boldsymbol{A}^T$ elements with their cofactors, see **DETERMINANTS**) and

$|\boldsymbol{A}|$ = determinant of $\boldsymbol{A}$.

Linear Algebra: Determinants and Inverse

DETERMINANTS

A *determinant of order n* consists of n^2 numbers, called the *elements* of the determinant, arranged in n rows and n columns and enclosed by two vertical lines.

In any determinant, the *minor* of a given element is the determinant that remains after all of the elements are struck out that lie in the same row and in the same column as the given element. Consider an element which lies in the jth column and the ith row. The *cofactor* of this element is the value of the minor of the element (if $i + j$ is *even*), and it is the negative of the value of the minor of the element (if $i + j$ is *odd*).

If n is greater than 1, the *value* of a determinant of order n is the sum of the n products formed by multiplying each element of some specified row (or column) by its cofactor. This sum is called the *expansion of the determinant* [according to the elements of the specified row (or column)]. For a second-order determinant:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

* If det|A| = 0 then A^{-1} does not have an inverse

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

For a third-order determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Ex: $A = \begin{bmatrix} -1 & 0 & 2 \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ Find $A^{-1} = \dfrac{adj(A)}{|A|}$

① Calc $|A| = -6$ (?)

② $Adj(A) = C^T$, C: matrix of cofactors

$C_{11} = \det\begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} \cdot (-1)^{i+j=1+1} = 2 \cdot 1 = 2$

$C_{12} = \det \underbrace{\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}}_{(-1 \cdot 1) - \cancel{(1 \cdot 0)}} (-1)^3 = -1(-1) = +1$

$C = \begin{bmatrix} 2 & 1 & -4 \\ -4 & 3 & -2 \\ 4 & 2 & 2 \end{bmatrix}$

$A^{-1} = \dfrac{adj(A)}{|A|} = \dfrac{C^T}{|A|} = \dfrac{\begin{bmatrix} 2 & -4 & 4 \\ 1 & 3 & 2 \\ -4 & -2 & 2 \end{bmatrix}}{|A|}$

Linear Algebra: Matrices

Which of the following statements regarding matrices is FALSE?

(A) $(\mathbf{A}^T)^T = \mathbf{A}$

(B) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

(C) $\begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \end{pmatrix}$

(D) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$

Example: Determinants

What is the determinant of the following 2×2 matrix?

$$\begin{pmatrix} 5 & 9 \\ 7 & 6 \end{pmatrix}$$

(A) -33 (B) -27 (C) 27 (D) 33

What is the determinant of the following matrix?

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

(A) 0 (B) 1 (C) 5 (D) 7

Fundamentals of Engineering: Mathematics Review

Example: Inverse Matrices

What is the inverse of the matrix $\mathbf{A}$?

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(A) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (C) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

(B) $\begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (D) $\begin{pmatrix} \cos\theta\sin\theta & 0 \\ 0 & \sin\theta\cos\theta \end{pmatrix}$